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Election results and the Sznajd model on Barabasi network

A.T. Bernardes[1,2], D. Stauffer[2,a], and J. Kertész[3]

[1] Departamento de Física, Universidade Federal de Ouro Preto, Campus do Morro do Cruzeiro, 35400-000, Ouro Preto-MG, Brazil
[2] Institute for Theoretical Physics, Cologne University, 50923 Köln, Germany
[3] Department of Theoretical Physics, Budapest University of Technology and Economics, Budafoki út 8, 1111, Budapest, Hungary

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Abstract. The network of Barabasi and Albert, a preferential growth model where a new node is linked to the old ones with a probability proportional to their connectivity, is applied to Brazilian election results. The application of the Sznajd rule, that only agreeing pairs of people can convince their neighbours, gives a vote distribution in good agreement with reality

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1 Introduction

Nowadays, it has been a matter of increasing interest [1] to apply the fundamentals of the theories of complex systems in many different disciplines, not only in physical sciences, but even in social sciences, from economy to education [2] or sociology. The main point is that social systems, like natural ones, are constituted of great number of individuals, which – generally – have local interactions between them. Sometimes, social networks behaviour can be determined also by the action of external actors, which might be mimicked by external fields in our model.

Elections are processes where many individuals interact between them. It is a dynamical convincing process, where we have at the same time the interaction between neighbours and external influence (political advertising, campaigns etc). In Brazil, in proportional elections (deputies or city councillors) the voters vote directly for the candidates and not for the parties. They can vote for a party, but it is not frequent. Some elections occur with a large number of voters: In some states or in the largest cities one has a number of voters in the order of magnitudes of millions or tens of millions. So, these elections are a social phenomenon which presents the basic characteristics of complex systems. One of these features is that they are scale-free phenomena. This feature has been observed by Costa-Filho *et al.* [3], who showed that the distribution of the number of votes obtained by different candidates for the 1998 elections in Brazil follow a power law distribution, with exponent $\simeq -1.0$. The same result can be obtained for the whole country or for other proportional elections, which shows that this is a very robust result.

Taking into account that elections are processes where a vote is supposed to be obtained as a result of convincing arguments, it can be compared with a physical process of clustering. However, contrary to the results obtained by Costa-Filho *et al.*, usual models of formation of clusters (as percolation, for instance [4]) may give exponents $\simeq -2.0$ (square lattice) for the numbers of clusters as a function of the cluster size.

Recently, we have introduced [5] a model for proportional elections. Our model is based on the Sznajd model proposed to simulate the process of formation of opinion. However, different from other models [6], where the influence flows inward from the border to the center (like in the majority games, where the site in the middle takes the state of the majority of neighbouring sites), in the Sznajd model [7–10] one has an outward flow of influence. It thus differs from *e.g.* bootstrap percolation [11] or other cellular automata (for a computational review see [12]) where the site in the center behaves according to a rule determined by its neighbours. Nevertheless, the dynamics of the Sznajd model is quite similar (except for isolated sites) to that of spinodal decomposition of the Ising model [13] at low temperatures, as was shown in [5]: Starting from a random distribution, large domains form where nearly all sites have the same state. Finally, one domain will cover the whole lattice. Thus to get the desired results we will

[a] e-mail: `stauffer@thp.uni-koeln.de`

look at intermediate times when there are still many different domains or correlated sites in the system.

In the Sznajd model, small sets of people influence the opinions of their nearest neighbours if and only if all people within the original set agree. On a chain, this set is a bond with two people at its ends [7,9]. On the square lattice with [10] or without [8] disorder, it can be such a bond or a plaquette of four neighbouring spins (people). This plaquette rule is called rule Ia in [8] and the bond rule, used in the present paper, is called rule IIa in [8]. Thus if all four plaquette members or two bond members share the same opinion, they convince their neighbours of this opinion; otherwise the neighbours remain unchanged.

However, as shown by Barabasi and Albert recently [14], social relations must be represented by networks instead of lattices. Networks of interactions (www, author's collaboration in scientific papers, actor's collaborations in films) show the common feature of scale-free behaviour. In order to represent this main feature, Barabasi and Albert introduced a model for evolving networks. Numerous papers used this model for a variety of purposes, *e.g.* [15]. Starting with few nodes (which may represent actors, authors, web sites) connected to each other, more and more nodes are added to the network, each node connecting to an already connected one, with the probability to connect to a node being proportional to the number of previous nodes which are already connected to it.

In Brazil the voting process is much more based on the relation between candidate/voter than on the parties. Thus, our idealized version of the voting process can omit the role of the parties. Another aspect is that it was clear for us that the candidates do not start with the same social weight. This determines the result of the elections, since candidates with more social visibility or better conditions to campaign are more likely to be elected. So, we have to introduce some differentiation between the candidates in the beginning of the simulation.

In this paper, we performed simulations on a three-dimensional version of the previous one-dimensional [7] and two-dimensional [8] Sznajd model. We combine it with a network model for elections based on the model of Bernardes *et al.* [5]. Unlike this first version [5] and its three-dimensional variant where a probability to convince had to be introduced, equation (1) below, in order to produce some differentiation between the candidates, with the Barabasi network the same result is obtained from the combination of the different number of neighbours of the nodes without this probability.

In the next section we present the models we have used, followed by the results. Both of these models use networks connecting the voters; one network is a simple cubic lattice, the other a Barabasi network. After that, we conclude.

2 Models and results

We have simulated two models in the present work. The first one is a 3d version of that simulated previously [5]. The second is a Barabasi network version.

2.1 Simple cubic lattice

In this work, we used a modified version of the Sznajd model (rule IIa in [8]): A pair of neighbours in agreement convinces its ten nearest neighbours to the same opinion. A cubic lattice of size $L \times L \times L$ represents the set of voters. A number $N_{\rm tot}$ of candidates, $N_{\rm tot} \ll L^3$, is fixed in the beginning of the simulation. The value $n = 1, 2, \ldots, N_{\rm tot}$ of a site S on the lattice will represent that this voter prefers that candidate n. The model has two different stages: First, we produce the initial condition and, after that, we perform the simulation of the electoral campaign (only voters can influence other voters, *a la* Sznajd). As in real elections, we do not wait for a kind of equilibrium state, but count the votes at some intermediate time. Basically what we are doing is the analysis during the transient time. As in real elections, the candidates have different initial chances of being voted for (representing more money for electoral campaigns, more initial visibility etc.). This is modelled by a probability $P_{\rm c}$ of convincing, calculated from the label n of the candidate

$$P_{\rm c} = (n/N_{\rm tot})^2. \tag{1}$$

It means that the higher is the label n of a candidate, the higher is the probability of convincing a voter.

In the first stage, we started with all the sites with value zero, meaning that there are no committed voters. Then, we visit all the sites exactly once, in random order. For each visit, we try to convince the voter to adopt a candidate, chosen at random. A random number r is generated and compared with $P_{\rm c}$. If $r \le P_{\rm c}$ the candidate is accepted by that voter. If the candidate convinces the voter, this voter tries to convince the neighbouring sites. Once again, we throw the dice and compare a new random number with $P_{\rm c}$. If successful, $r \le P_{\rm c}$, the voter will try to convince the neighbourhood as follows: We check all the six neighbouring sites; for each that has the same value of the candidate chosen before, all the ten neighbouring sites of this bond of two sites will assume the same value (as in the usual Sznajd prescription). If nobody has chosen the same candidate, only the originally selected voter is committed to this candidate.

In the second stage, a usual Sznajd process is performed without using the complication from the probability $P_{\rm c}$. (We thus assume all voters to be equal and restrict the probability $P_{\rm c}$ to describe the convincing power of the candidates only.) We go to random sites on the lattice. A neighbouring site is chosen at random and we check if the two sites have the same value (they prefer the same candidate). In that case, all the ten neighbours change to vote in that candidate.

Figure 1 shows, just as in real life [3], deviations from a simple power law for both very large numbers of votes and very small numbers. In between, however, the simulations are compatible with the hyperbolic law

$$N(v) \propto 1/v \tag{2}$$

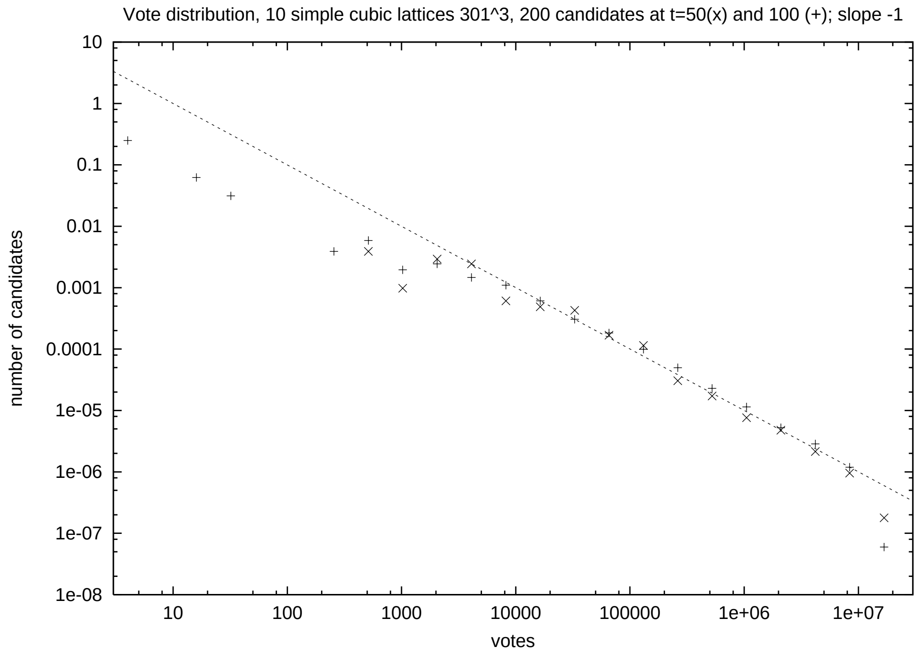

Fig. 1. Distribution $N(v)$ of the number of candidates getting v votes each on the simple cubic lattice, after 50 and 100 iterations. Election of 200 candidates by 27 million voters.

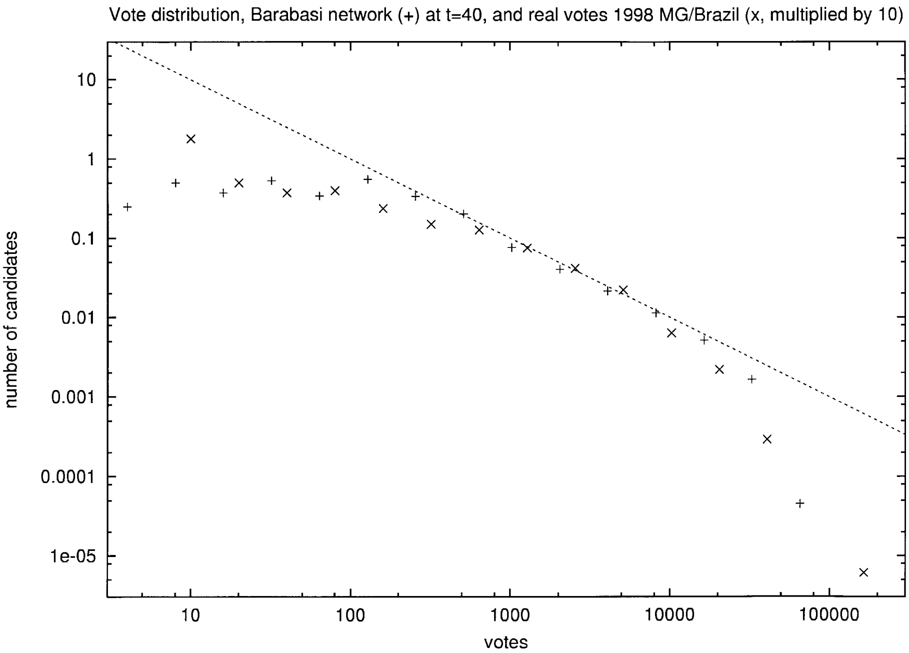

Fig. 2. Distribution $N(v)$ for half a million nodes on the Barabasi network, where each previously added node bonds to five previously added nodes. Election of 1000 candidates (+). The number of votes in real elections (×: state of Minas Gerais in Brazil 1998) is multiplied by ten for better comparison.

observed in reality for the number N of candidates having v votes each. (Here and in Fig. 2 below the bin size for v increases by a factor 2 for each consecutive bin.)

[In [5], for the square lattice and assumption (1), *two* exponents are fitted onto the data: One for the distribution after the first stage, and one for the distribution during the second change. When assumption (1) is generalized from $\propto n^2$ to $\propto n^x$, then the first exponent depends strongly on x (which can be explained by a simple analytical relation between this exponent and x), while the second exponent depends much less on x.]

2.2 Network version

In this model, we first create a network of interacting nodes by using the basic Barabasi-Albert prescription. We fix an initial number of nodes, each one connected to the others. In the present work, the minimum number of connections of a node is $m = 5$. So, in the beginning of the simulation we have 6 nodes, in order that each one can be connected to the 5 others. After that, more and more nodes are added to the network. A new node has a probability to be connected to a previous node proportional to the number of nodes that are already connected to this previous node. Thus the growth probability at any existing node is proportional to the number of nodes already connected to it. We will no longer need assumption (1) and have replaced this assumption and the regular lattice by the Barabasi network without such an assumption.

After preparing the network, we start with the election process, which is now different from that in the previous three-dimensional model. The first step is the distribution of candidates. Again, the state of a node, that means, the value n of a node on the network, represents that this voter has given the preference to that candidate n. Thousand candidates are distributed at random, disregarding the number of connections of a node, *i.e.* we pick a node at random from the half million nodes to which we let the network grow, and then a candidate at random. Now, the campaign starts. At each time step we visit all the nodes. For each node, we have the following process:

- If a node i has already selected preference for a candidate, we choose a connected node j at random. If node i has no candidate ($n = 0$), we go to another randomly selected node.
- If node j has the same candidate as node i, they try to convince all the nodes connected with them. The probability to convince others for each of the two nodes is now inversely proportional to the time-independent numbers of nodes connected with it, meaning that each node convinces – on average – one other node at each process.
- If node j has no candidate, node i tries to convince it to accept its own candidate, with the same probability as described above.
- If node j has a different candidate from node i, we skip to another node i.

Again, as described above for the 3d version, we do not wait for a equilibrium state. It is important to mention that, different from a square lattice, where an equilibrium state is reached in a time proportional to the number of sites, in the network it is reached rapidly, after about 10^2 iterations. In both cases, in final equilibrium all the sites have the same state.

Figure 2 shows that again except for the smallest and the largest numbers v of votes, the hyperbolic law (2) is obeyed well at intermediate times $t = 40$.

3 Summary

Whether we simulate the election process on a square lattice, a simple cubic lattice or a Barabasi network, we recover the same hyperbolic law as found in real elections. Our simulations on a Barabasi network have the advantage that we no longer need assumption (1) for the purpose of getting a realistic vote distribution with decay exponent 1 in the center.

Our study has shown that the hyperbolic law observed empirically is a rather robust consequence of our modifid Sznajd model, since we found it first on the square lattice [5] and now on both the cubic lattice and the Barabasi network. Either we use a regular lattice and assumption (1), or we use the Barabasi network without assumption (1); the final results are similar. The fact that the hyperbolic law is observable on the Barabasi network, which is a more realistic model of social interactions than the lattices, provides evidence that the Sznajd model may well capture important aspects of the voting mechanism. The advantage of using the Barabasi network instead of regular lattices is not only that it is more realistic but also that we can drop assumption (1) which is a kind of fine tuning the system to criticality. Of course, the Barabasi model, and the related assumptions are also rules (as we have rules when constructing a lattice too) but the difference is similar to what we have for usual and self-organized criticality: Assumption (1) puts in some exponent at the beginning, through $(n/N_{\text{tot}})^2$, on which the final exponent depends somewhat (see end of Sect. 2.1), while the Barabasi growth process leads by itself to a power-law distribution of the number of connections for a node, and combined with the Sznajd model gives the desired vote distribution with its intermediate power law.

Moreover, rule (1) was introduced ad hoc to explain the election results, while the rules of [14] were stated before, independent of the present application. We are not aware of other voter models[16] explaining the hyperbolic law found empirically in [3].

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